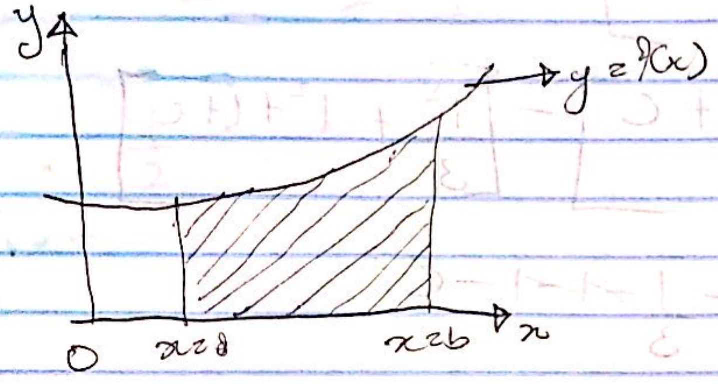
Integration Application

To find the area bounded by the curve $y = f(x)$ with the ordinates at $x = a$ and $x = b$



There is no formula for this calculation since the shape of curve depends on the function $f(x)$

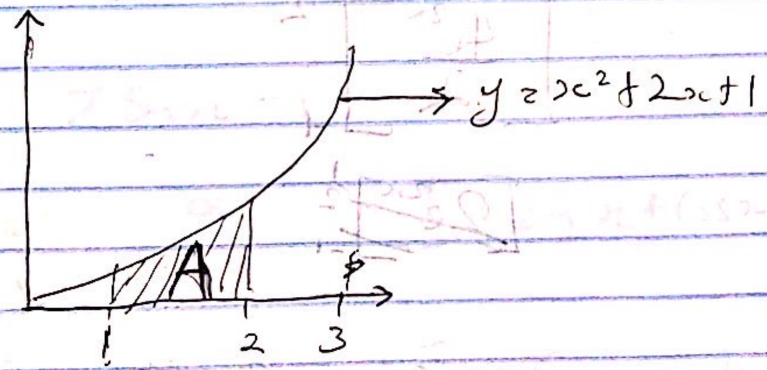
Integrating

$$A_x = \int_a^b f(x)\,dx$$

$$A_x = F(x) + C$$

Find the area under the curve, $y = x^2 + 2x + 1$ between $x = 1$ and $x = 2$

Solution



$$A = \int_1^2 y\,dx = \int_1^2 (x^2 + 2x + 1)\,dx$$

$$= \left[\frac{x^3}{3} + x^2 + x + C\right]$$

$$\left[\frac{(2)^3}{3} + 2^2 + 2 + C\right] - \left[\frac{1^2}{3} + 1^2 + 1 + C\right]$$

$$= \frac{8}{3} + 4 + \cancel{2} + \cancel{C} - \frac{1}{3} - \cancel{1} - 1 - \cancel{C}$$

$$\frac{8}{3} - \frac{1}{3} + 4 = \frac{19}{3} = 6\frac{1}{3} \text{ unit}^2$$

DEFINATE INTEGRALS

An integral with limits is called definite integral with a definite integral

The constant of integration C is cancelled out

Ex.2

Evaluate $\int_{-1}^{1/2} 4e^{2x}\,dx$

$$\left[\frac{4e^{2x}}{2}\right]_{-1}^{1/2}$$

$$\left[\cancel{2e^{2x}}\right]_{-1}^{1/2}$$

$$\cancel{2}\,6 \quad \cancel{2(-1)^{2x} - 2\left(\tfrac{1}{2}\right)^{2x}}$$

$$\cancel{-2^{2x} - 1^{2x}}$$

$$4\left[\frac{e^{1}}{2}-\frac{e^{-2}}{2}\right]$$

$$\frac{4}{2}\left[e-e^{-2}\right]$$

$$=2e-2e^{-2}$$

Exc. 3

Evaluate $\int_{0}^{\frac{\pi}{2}} x\cos x\,dx$

$$\int u\,dv = uv-\int v\,du$$

$\frac{du}{dx}=1$

$u=x \qquad du=dx$

$dv=\cos x \qquad v=\int \cos x\,dx$

$v=\sin x$

$$\int x\cos x\,dx = 1\times\sin x-\int \sin x\,dx$$

$$x\sin x-(-\cos x)$$

let $u=x$

dv

$$\left[x\sin x+\cos x\right]_{0}^{\frac{\pi}{2}}$$

$$\left[\frac{\pi}{2}\sin\frac{\pi}{2}+\cos\frac{\pi}{2}\right]-\left[0\sin 0+\cos 0\right]$$

$$=\left[\frac{\pi}{2}+0\right]-\left[0+1\right]$$

$= \frac{1}{2} - 1$

Ex. 4

$$\int_1^2 (2x-3)^4\,dx = \left[\frac{(2x-3)^5}{10}\right]_1^2$$

$$\frac{1}{10}\left[[2(2)-3]^5 - [2(1)-3]^5\right]$$

$$\frac{1}{10}\left[1^5 - (-1)\right]$$

$$= \frac{1}{10}(2) = \frac{1}{5}$$

$\cancel{u = 2x-3}$

$\cancel{\frac{du}{dx} = 2}$

$\cancel{dx = \frac{du}{2}}$

$\cancel{\int_1^2 (2x-3)}$ $\quad$ $\cancel{\int_1^2 u^4 \frac{du}{2}}$

$\cancel{\frac{1}{2}\int_1^2 u^4\,du}$

$\cancel{\frac{1}{2}\int_1^2 u^2}$ $\quad$ $\cancel{\frac{1}{2}\int_1^2}$

$$\int_1^2 z^4\,dx = \int_1^2 z\left(\frac{dx}{dz}\right)dz$$

$$\frac{dz}{dx} = 2,\ \frac{dx}{dz} = \frac{1}{2}$$

$$= \frac{1}{2}\int z^4\,dz = \frac{1}{2}\left[\frac{z^5}{5}\right] = \frac{z^5}{10}$$

$$\left[\frac{(2x-3)^5}{10}\right]_1^2$$

2 Parametric Equations

Ex. 1

If a curve has parametric equations $x = at^2$, $y = 2at$
Find the area bounded by the curve the ordinates at $t=1$ and $t=2$

Solution

$$A = \int_a^b f(x)\,dx \quad = \quad A = \int_a^b y\,dx$$

$$A = \int_a^b 2at\,dx$$

$x = at^2$

$\frac{dx}{dt} = 2at, \quad dx = 2at\,dt$

$$A = \int_a^b 2at \cdot 2at\,dt$$

$$= \int_a^b 4a^2 t^2\,dt = 4a^2 \int_1^2 t^2\,dt$$

$$= 4a^2 \left[\frac{t^3}{3}\right]_1^2$$

$$= 4a^2\left[\frac{2^3}{3} - \frac{1^3}{3}\right]$$

$$= 4a^2\left[\frac{8}{3} - \frac{1}{3}\right] = 4a^2\left[\frac{7}{3}\right] = \frac{28a^2}{3}$$

Ex. 2

A $\quad x = \theta - \sin\theta$

$\quad y = 1 - \cos\theta$

find the area under the curve between $\theta = 0$ and $\theta = \pi$

$$\int_a^b y \, dx$$

$$\int_a^b 1 - \cos\theta \, dx$$

$$\int_0^{\pi} 1 - \cos\theta \, dx$$

$x = \theta - \sin\theta$

$$\frac{dx}{d\theta} = 1 - \cos\theta$$

$$dx = (1 - \cos\theta)\, d\theta$$

$$\int_a^b (1-\cos\theta)(1-\cos\theta)\, d\theta$$

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$$\int_{0}^{3}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

$$\int_{0}^{\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

$\cancel{2\cos^2\theta}$ $\quad \cos^2\theta+\cos^2\theta = 1+\cos 2\theta$

$$2\cos^2\theta = 1+\cos 2\theta$$

$$\therefore \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\int_{0}^{\pi}\left(1-2\cos\theta+\frac{1}{2}+\frac{\cos 2\theta}{2}\right)d\theta$$

$$= \left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi}$$

$$= \left[\pi - 2\sin\pi + \frac{\pi}{2} + \frac{\sin 2\pi}{4}\right] - \left[0 - 2\sin 0 + \frac{0}{2} + \frac{\sin 2\times 0}{4}\right]$$

$$= \left[\pi - 0 + \frac{\pi}{2} + 0\right] - \left[0 - 0 + 0 + 0\right]$$

$$= \pi + \frac{\pi}{2} = \frac{3\pi}{2}\ \text{Unit}^2$$

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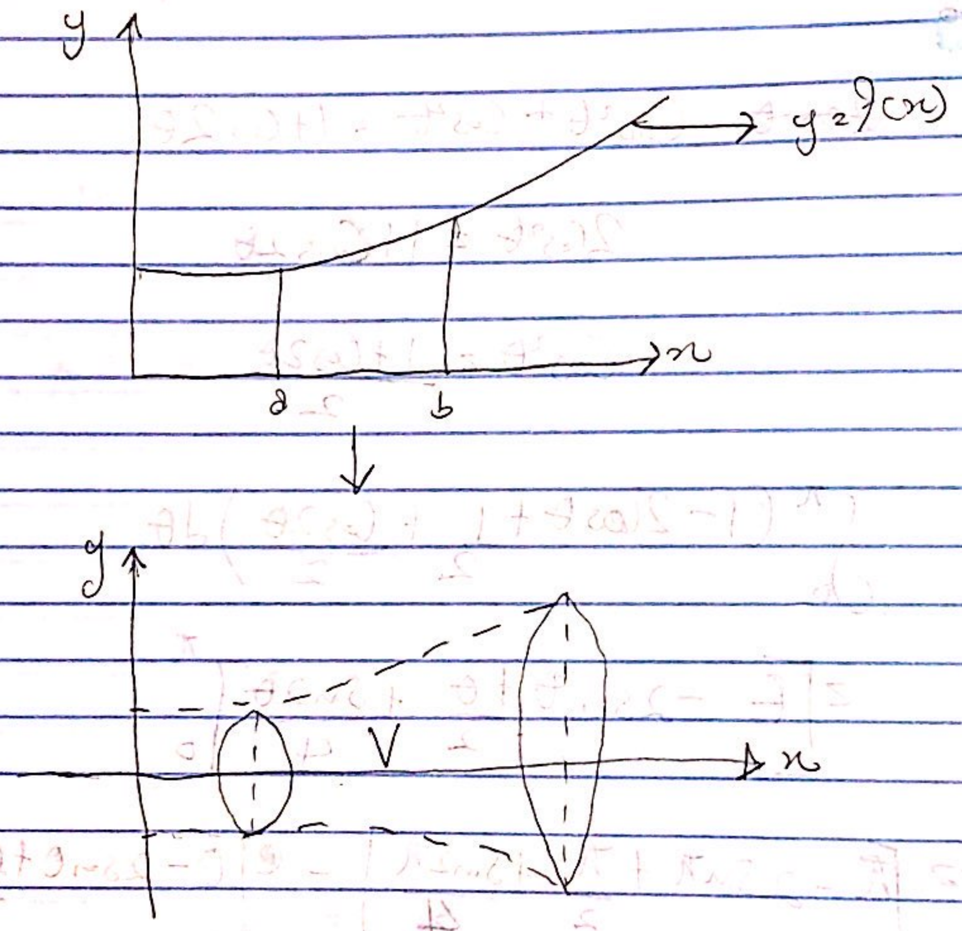
Volumes of Solids of Revolution

If a plane figure bounded by the curve $y = f(x)$, the x-axis, the ordinate at $x = a$, $x = b$ rotates through a complete revolution about the x-axis, it would generate a ~~symmetrical~~ symmetrical solid about OX



Let the volume, V of the generated solid be,

$$V = \int_a^b \pi y^2 \, dx$$

Find the volume generated which the plane figure bounded by $y = 5\cos 2x$, ordinates $x = 0$ and $\frac{\pi}{4}$

$$\int_a^b \pi y^2\,dx$$

$$V = \int_a^b \pi(5\cos 2x)^2 \cdot dx$$

$$\pi \int_a^b 25\cos^2 2x \cdot dx$$

$$25\pi \int_0^{\pi/4} (\cos^2 2x)\,dx$$

$$25\pi \int_0^{\pi/4} \frac{1+\cos 4x}{2}\,dx$$

$$\frac{25\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{4}}$$

$$\frac{25\pi}{2}\left\{\left[\frac{\pi}{4} + \frac{\sin\pi}{4}\right] - \left[0 + \frac{\sin 0}{4}\right]\right\}$$

$$= \frac{25\pi^2}{8} \text{ unit}^3$$

The parameter equation of a curve are $x = 3t^2$, $y = 3t - 3^2$. Find the volume generated when the plane figure bounded by the curve, the x axis and the coordinate corresponding to $t = 0$ and $t = 2$, rotates about the x-axis.