DIFFERENTIATION

TECHNIQUES OF DIFFERENTIATION

STANDARD DERIVATIVES

$y = f(x)$	$\frac{dy}{dx}$
x^n	nx^{n-1}
e^x	e^x
e^{kx}	ke^{kx}
a^x	$a^x \cdot \ln a$
$\ln x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \cdot \ln a}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \cdot \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cdot \cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$+\sinh x$
$\sin(kx)$	$k\cos(kx)$
$\cos(kx)$	$-k\sin(kx)$

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REVIEW OF BASIC DIFFERENTIATION TECHNIQUES

① CHAIN RULE

It states that if y is a function of u ~~and v~~ and u is a function of x then, the derivative $\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$

② FOR INSTANCE : $\sin(2x+5)$ is a function of the angle $2x+5$ since the value of the sine depends on the value of $2x+5$ that is $\sin(2x+5)$ is a function of $2x+5$ but $2x+5$ itself is also a function of x ($f(x)$)

Hence $\sin(2x+5)$ is a function of a function of x and such expression is reffered to as function of a function

Example 1

differentiate the function

$$y = \cos(5x+4)$$

Solution

let $u = 5x+4$

$$\frac{du}{dx} = 5$$

$$y = \cos u$$

$$\frac{dy}{du} = -\sin u$$

$$\frac{dy}{dx} = \frac{du}{dx} \cdot \frac{dy}{du}$$

$$5 - \sin u = -5\sin u$$

$$\frac{dy}{dx} = -5\sin(5x+4)$$

~~$y = \ln(3-4$~~

$$y = \ln(3 - 4\cos x)$$

$$y = \ln u$$

$$u = 3 - 4\cos x$$

$$\frac{du}{dx} = 4\sin x$$

$$y = \ln u$$

$$\frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{1}{u} \times 4\sin x$$

$$\frac{dy}{dx} = \frac{4\sin x}{u} = \cancel{4\sin x u}$$

$$4u^{-1}\sin x$$

$$\frac{4\sin x}{(3-4\cos x)} = 4\sin x(3-4\cos x)^{-1}$$

17 ~~$y = \log_{10}$~~ $y = \log_{10}(2x-1)$

$$\frac{dy}{dx} = \log_{10} \partial$$

$$\partial = 2x - 1$$

$$\frac{d\partial}{dx} = 2$$

$$\frac{dy}{dx} = \frac{1}{a \ln 10}$$

$$\frac{1}{a \times 2.3026}$$

$$\frac{dy}{dx} = \frac{1}{2.3026a} \times 2$$

$$\frac{dy}{dx} = \frac{2}{2.3026a} = \frac{2}{2.3026(2x-1)}$$

$$\frac{2}{\ln 10 (2x-1)}$$

Assignment

Faraday's law states that the electromotive force E induced by N turns of a coil with a flux ϕ ~~of a~~ passing through it, is given by $\boxed{E = -N\frac{d\phi}{dt}}$

If ~~the~~ ϕ

$$\phi = K \sin(2\pi f t)$$

when K and f are constants

Find E

(2) Product Rule

If $y = UV$, where U and V are functions of x then $\frac{dy}{dx} = U\frac{dv}{dx} + V\frac{du}{dx}$

Example I.

Given a function $y = x^3 \sin 3x$

find $\frac{dy}{dx}$!

$\sin 3x$

x^3 $3\cos 3x$

$3x^2$

$9x^2 \cos 3x$

$u = x^3$

$\frac{du}{dx} = 3x^2$

$x \cos 3x$

$\frac{dy}{dx}$ $\quad v = 3 \sin 3x$

$\frac{dv}{dx} = 3 \cos 3x$

$\cancel{x^3 \; 3x^3 \cos 3x + x^3}$

$3x^3 \cos 3x + 3x^2 \sin 3x$

$3x^2(x \cos 3x + \sin 3x)$

$\cancel{\frac{dy}{dx} = x^3 \cdot 3\cos}$

If $y = x^2 \ln \sinh x$, find $\frac{dy}{dx}$

$y = x^2 \ln t$

$t = \sinh x$

$\frac{dt}{dx} = \cosh x$

$\frac{dy}{dx} = 2x \ln t + \frac{x^2}{t}$

$\frac{dy}{dx} = 2x \ln t \cosh x + \frac{x^2 \cosh x}{t}$

$\frac{dy}{dx} = 2x \ln \sinh x \cosh x + \frac{x^2 \cosh x}{\sinh x}$

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$$\frac{dy}{dx} = 2x\ln(\sinh x)\cancel{\cosh x} + x^2\frac{\cosh x}{\sinh x}$$

$$\frac{dy}{dx} = 2x\ln\sinh x\,\cancel{\cosh x} + x^2\coth x$$

let $u = x^2$ and $V = \ln\sinh x$

$u = x^2$

$$\frac{du}{dx} = 2x$$

$$\frac{dV}{dx} = ?$$

$V = \ln w$

$$\frac{dw}{dx} = \cosh x$$

$$\frac{dV}{dw} = \frac{1}{w}$$

$$\frac{dy}{dx} = \frac{x^2\cosh x}{\sinh x} + 2x\ln\sinh x$$

$$\frac{dy}{dx} = x^2\coth x + 2x\ln\sinh x$$

$$\frac{dy}{dx} = x(x\coth x + 2\ln\sinh x)$$

If $y = x^3 \sin 5x$, find $\frac{dy}{dx}$ Soln

$u = x^3$

$\frac{du}{dx} = 3x^2$ $\quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$v = \sin 5x$

$\frac{dv}{dx} = 5\cos 5x$

$5x^3 \cos 5x + 3x^2 \sin 5x$

$x^2(5x\cos 5x + 3\sin 5x)$

3 QUOTIENT RULE

If you and v are functions of x and $y = \frac{u}{v}$

Then, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = \frac{\sin 3x}{x+1}$ find $\frac{dy}{dx}$

$u = \sin 3x = \frac{du}{dx} = 3\cos 3x$

$v = x+1 = \frac{dv}{dx} = 1$

$\frac{(x+1) \times 3\cos 3x - 1 \times \sin 3x}{(x+1)^2}$

$$\frac{3\cos 3x(x+1) - \sin 3x}{(x+1)^2}$$

$$\frac{3x + 3\cos 3x - \sin 3x}{(x+1)^2}$$

Example 2

Prove that the derivative of $\tan x$ is $\sec^2 x$

If $y = \tan x$

$y = \tan x$

$y = \frac{\sin x}{\cos x}$ $= u$, $= v$

~~$y = \tan x$~~

~~$y + \Delta y = \tan(x + \Delta x)$~~

~~$\Delta y = \tan(x + \Delta x) - y$~~

~~$\Delta y = \tan(x + \Delta x) - \tan x$~~

~~$\Delta y = \tan x + \tan \Delta x - \tan x$~~

$\frac{du}{dx} = \cos x$, $\frac{dv}{dx} = -\sin x$

$\frac{dy}{dx} = \cos x \cdot \cos x - (\sin x \cdot -\sin x)$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$

Recall = $\cos^2\theta + \sin^2\theta = 1$

so $= \frac{1}{\cos^2 x} = \sec^2 x$

$\therefore$ This implies that the derivatives $= \sec^2 x$

(4) ~~Technique~~ Logarithmic Differentiation

The previously discussed product and quotient differentiation rules are used when there are just two function that is $\frac{U}{V}$ or UV but when there are more than two functions in any arrangement top or bottom the derivative is first found by logarithmic differentiation

For instance

$y = \frac{UV}{W}$, when U, V, W and also y are all functions of x, then you take log to base of the given function

$$\boxed{\log_e = \ln}$$

$$\ln y = \ln U + \ln V - \ln W \quad (1)$$

differentiate both sides of the equation in respect to x, where U, V, W and y are all functions of x

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{U} \times \frac{du}{dx} + \frac{1}{V} \times \frac{dV}{dx} - \frac{1}{W} \times \frac{dw}{dx} \quad (2)$$

multiply equation (2) by y to make $\frac{dy}{dx}$ ~~=~~ ~~y~~ the subject of the equation

$$\frac{dy}{dx} \cancel{= \frac{1}{y} \times \frac{dy}{dx}}$$

$$\frac{dy}{dx} = y\left[\frac{1}{u} \times \frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx}\right]$$

$$\frac{dy}{dx} = \frac{uv}{w}\left[\times\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} - \frac{1}{w}\frac{dw}{dx}\right]$$

Example 1

If $y = \frac{x^2 \sin x}{\cos 2x}$, find $\frac{dy}{dx}$

$$\ln y = \ln(x^2) + \ln(\sin x) - \ln(\cos 2x)$$

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{x^2} \times 2x + \frac{1}{\sin x} \times \cos x - \frac{1}{\cos 2x} \times \frac{-2\sin 2x}{1}$$

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{2}{x} + \frac{\cos x}{\sin x} + \frac{2\sin 2x}{\cos 2x}$$

$$\frac{1}{y} \times \frac{dy}{dx} = 2x^{-1} + \cot x + 2\tan 2x$$

$$\frac{dy}{dx} = y\left[2x^{-1} + \cot x + 2\tan 2x\right]$$

$$\frac{dy}{dx} = \frac{x^2 \sin x}{\cos 2x}\left[2x^{-1} + \cot x + 2\tan 2x\right]$$

$$E = -N\frac{d\phi}{dt}$$

$$\phi = K \sin(2\pi f t)$$

K and f are Constants Determine E.

Solution

let $= 2\pi f t$

$\phi = K \sin u$

$u = 2\pi f t$

$$\frac{du}{dt} = 2\pi f$$

$$\frac{d\phi}{du} = K \cos u$$

$$\frac{d\phi}{dt} = \frac{d\phi}{du} \times \frac{du}{dt}$$

$$\frac{d\phi}{dt} = K\cos u \times 2\pi f$$

$$\frac{d\phi}{dt} = 2\pi f K \cos u$$

$$2\pi f K \cos(2\pi f t)$$

$$E = -2\pi f K N \cos(2\pi f t)$$

Ex. 2

If the function $y = x^4 e^{3x} \tan x$

find $\frac{dy}{dx}$

$$y = x^4 e^{3x} \tan x$$

$$\ln y = \ln x^4 + \ln e^{3x} + \ln \tan x$$

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{x^4} \times 4x^3 + \frac{1}{e^{3x}} \times \frac{3e^{3x}}{1} + \frac{1}{\tan x} \times \sec^2 x$$

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{4}{x} + 3 + \frac{\sec^2 x}{\tan x}$$

$$\frac{1}{y} \times \frac{dy}{dx} = 4x^{-1} + 3 + \frac{\sec^2 x}{\tan x}$$

$\times$ by y

$$\frac{dy}{dx} = y\left[4x^{-1} + 3 + \frac{\sec^2 x}{\tan x}\right]$$

$$\frac{dy}{dx} = x^4 e^{3x} \tan x \left[4x^{-1} + 3 + \frac{\sec^2 x}{\tan x}\right]$$

Ex3

1) $y = \frac{e^{4x}}{x^3 \cosh 2x}$

find $\frac{dy}{dx}$

$$\ln y = \ln e^{4x} - \ln x^3 + \cosh 2x$$

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{1}{e^{4x}} \times 4e^{4x} - \frac{1}{x^3} + 3x^2 + \frac{1}{\cosh 2x} \times 2\sinh 2x$$

$$\frac{1}{y} \times \frac{dy}{dx} = 4 - \frac{3}{x} + \left(-\frac{2\sinh 2x}{\cosh 2x}\right)$$

$$\frac{1}{y} \times \frac{dy}{dx} = 4 - 3x^{-1} - 2\tanh 2x$$

$$\frac{dy}{dx} = y\left[4 - 3x^{-1} - 2\tanh 2x\right]$$

$$\frac{dy}{dx} = \frac{e^{4x}}{x^3 \cosh 2x}\left[4 - 3x^{-1} - 2\tanh 2x\right]$$

5

IMPLICIT FUNCTIONS

Consider a function $y = x^3 - 4x + 2$ when y is completely defined in terms of x. y is called an explicit function of x

Ex 1

If ~~$y = x^2 + y^2$~~ $x^2 + y^2 = 25$

~~find~~ find ~~y~~ $\frac{dy}{dx}$

Soln

$$2x + 2y\frac{dy}{(dx)} = 0$$

$$2y\frac{dy}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Ex. 2

If $x^2 + y^2 - 2x - 6y + 5 = 0$

find $\frac{dy}{dx}$ and $\frac{d^2x}{dy^2}$ at $x = 3, y = 2$

Soln

$$2x + 2y\frac{dy}{dx} - 2 - 6\frac{dy}{dx} = 0$$

$$2x - 2 + \frac{dy}{dx}(2y - 6) = 0$$

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$$\frac{dy}{dx}(2y-6) = 2-2x$$

$$\frac{dy}{dx} = \frac{2-2x}{2y-6} = \frac{2(1-x)}{2(y-3)}$$

$$\frac{dy}{dx} = \frac{1-x}{y-3}$$

$\frac{dy}{dx}$ @ $x=3,\ y=2$

$$\frac{dy}{dx} = \frac{1-3}{2-3} = \frac{-2}{-1} = 2$$

$\therefore \frac{dy}{dx}$ @ $x=3, y=2 \Rightarrow 2$

Solution

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right]$$

$$\frac{dy}{dx} = f'$$

$$\frac{d^2y}{dx^2} = f''$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{1-x}{y-3}\right]$$

let $U = 1-x$
$V = y-3$

$$\frac{du}{dx} = -1, \frac{dv}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$$

$$\frac{(y-3)(-1) - \frac{dy}{dx}(1-x)}{(y-3)^2}$$

$$\frac{-y+3 - (1-x)\frac{dy}{dx}}{(y-3)^2}$$

$x=3$ $y=2$ $\quad -2$

$$\frac{d^2y}{dx^2} = \frac{(-2+3) - (1-3)\frac{dy}{dx}\cdot 2}{(2-3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{+1+4}{1} = 5$$

$\frac{d^2y}{dx^2}$ @ $x=3, y=2 \Rightarrow \underline{\underline{+5}}$

Example 3 if $x^3+y^3+3xy^2=8$ find $\frac{dy}{dx}$ Resolve Later

$x^3+y^3+3xy^2=8$

$3x^2+\cancel{3x^2\frac{dy}{dx}}+6xy+\cancel{x}\ 0x^2+3y^2\frac{dy}{dx}+3y^2+6xy\frac{dy}{dx}=0$

$\cancel{3x^2\frac{dy}{dx}}+3y^2\frac{dy}{dx}=-3x^2-6xy$

$3x^2+3y^2=(-3y^2-6xy)\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{3x^2+3y^2}{-3y^2-6xy}$

$\frac{dy}{dx}=\frac{3(x^2+y^2)}{-3y(y+2x)}$

$\frac{dy}{dx}=\frac{\cancel{3}(x+y)^2}{\cancel{3}y(y+2x)}$

$\frac{dy}{dx}=\frac{(x+y)^2}{y(y+2x)}$

$$3x^2+3y^2\frac{dy}{dx}+3y^2+6xy\frac{dy}{dx}=0$$

$$3x^2+3y^2+\frac{dy}{dx}(3y^2+6xy)=0$$

$$\frac{dy}{dx}(3y^3+6xy) = -(3x^2+3y^2)$$

$$\frac{dy}{dx} = \frac{-3(x^2+y^2)}{3(y^2+2xy)}$$

$$\frac{dy}{dx} = \frac{-(x^2+y^2)}{(y^2+2xy)}$$

Ex

17 $x^2+2xy+3y^2=4$

Find $\frac{dy}{dx}$

$$2x+2y+2x\frac{dy}{dx}+\cancel{3}\,6y\frac{dy}{dx}=0$$

$$2x\frac{dy}{dx}+6y\frac{dy}{dx} = -2x-2y$$

$$\frac{dy}{dx}(2x+6y) = -2(x+y)$$

$$\frac{dy}{dx} = \frac{-2(x+y)}{2x+6y} \Rightarrow \frac{-2(x+y)}{2(x+3y)}$$

$$= \frac{-(x+y)}{x+3y}$$

PARAMETRIC EQUATIONS

For some problems it is more convient to represent a function by expressing x and y seperately interms of a third independent variable

$$y = \cos 2t$$
$$x = \sin t$$

The third variable, t is called a parameter while the x and y expressions are called parametric equation ~~have~~ however we still need to find the derivatives of the function in respect to x

Ex. 1, If $y = \cos 2t$, $x = \sin t$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$y = \cos 2t$$

$$\frac{dy}{dt} = -2\sin 2t$$

$$x = \sin t$$

$$\frac{dx}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \cancel{2\cos}\ \frac{-2\sin 2t}{\cos t}$$

$$\frac{dy}{dx} = \cancel{2\tan 2t}$$

Trigonometry Rule

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta + \cos^2\theta = 1$$

$$\frac{dy}{dx} \quad \frac{-2(2\sin t\cos t)}{\cos t}$$

$$\frac{dy}{dx} = -4\sin t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(-4\sin t)\cdot\frac{dx}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-4\sin t) \qquad \frac{d}{} \quad -4\cos t\cdot\frac{1}{\cos t}$$

$$\frac{dy}{dt} = -4\cos t\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx} = -4\cos t\cdot\frac{1}{\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{-4\cos t}{\cos t}$$

$$1\times\frac{dt}{dx}$$

$$1\div\frac{dx}{dt}$$

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$$\frac{dx}{dt} = \cos t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \cancel{2\cos} \; \frac{-2\sin 2t}{\cos t}$$

$$\frac{dy}{dx} = \cancel{-2\tan 2t}$$

Trigonometry Rule

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\sin 2\theta + \cos^2\theta = 1$$

$$\frac{dy}{dx} \; \frac{-2(2\sin t\cos t)}{\cos t}$$

$$\frac{dy}{dx} = -4\sin t$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(-4\sin t)\cdot\frac{dx}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-4\sin t) \qquad \frac{d}{} \quad -4\cos t\cdot\frac{1}{\cos t}$$

$$\frac{dy}{dt} = -4\cos t\cdot\frac{dt}{dx}$$

$$\frac{dy}{dx} = -4\cos t\cdot\frac{1}{\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{-4\cos t}{\cos t}$$

$$1\times\frac{dt}{dx}$$

$$1\div\frac{dx}{dt}$$

$\frac{d}{dx}$ Cosh x = Sinh x

$$y = \frac{e^{4x}}{x^2 \operatorname{Cosh} 2x}$$

~~$\ln y = \ln e^{4x}$~~

$$\ln y = \ln e^{4x} - [\ln x^3 + \ln \operatorname{Cosh} 2x]$$

$$\frac{1}{y} \times \frac{dy}{dx} = \frac{4e^{4x}}{e^{4x}} - \frac{3x^2}{x^3} - \frac{2 \operatorname{Sinh} 2x}{\operatorname{Cosh} 2x}$$

$$\frac{dy}{dx} = \frac{e^{4x}}{x^3 \operatorname{Cosh} 2x}\left[4 - \frac{3}{x} - 2\tanh 2x\right]$$

Example 2.17

$$y = 3\operatorname{Sin}\theta - \operatorname{Sin}^3\theta$$

$$x = \operatorname{Cos}^3\theta$$

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$$\frac{dy}{d\theta} = 3\operatorname{Cos}\theta - 3\operatorname{Sin}^2\theta \operatorname{Cos}\theta$$

$$\frac{dx}{d\theta} = -3\operatorname{Cos}^2\theta \operatorname{Sin}\theta$$

$$\frac{dy}{d\theta} = 3\operatorname{Cos}\theta(1 - \operatorname{Sin}^2\theta)$$

$\operatorname{Sin}^2\theta + \operatorname{Cos}^2\theta = 1 \Rightarrow \operatorname{Cos}^2\theta = 1 - \operatorname{Sin}^2\theta$

$$\frac{dy}{d\theta} = 3\operatorname{Cos}\theta(\operatorname{Cos}^2\theta)$$

$$\frac{dy}{dx} = \frac{\cancel{3}\,\cancel{\operatorname{Cos}\theta} \cdot \operatorname{Cos}^2\theta}{-\cancel{3}\operatorname{Sin}\theta\,\cancel{\operatorname{Cos}\theta}}$$

$$\frac{dy}{dx} = \frac{3\cos\theta \cdot \cos^2\theta}{-3\sin\theta\cos^2\theta}$$

$$= \frac{-\cos\theta}{\sin\theta} = -\cot\theta$$

$$\frac{dy}{dx} = -\cot\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\cot\theta)$$

$$\Rightarrow \frac{d}{d\theta}\left(-\cot\theta\right) \cdot \frac{d\theta}{dx}$$

$$= -(-\operatorname{cosec}^2\theta) = \frac{1}{-3\sin\theta\cos^2\theta}$$

$$= \frac{\operatorname{cosec}^2\theta}{-3\sin\theta\cos^2\theta} \Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{3\sin^3\theta\cos^2\theta}$$

$$\operatorname{cosec}^2\theta = \frac{1}{\sin^2\theta}$$

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Ex 3 If $x = a(\cos\theta + \theta\sin\theta)$
$y = a(\sin\theta - \theta\cos\theta)$

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

$\frac{dx}{d\theta} = a$

$u = \cos\theta + \theta\sin\theta$

$\frac{du}{d\theta} = -\sin\theta + \sin\theta + \theta\cos\theta$

$\cancel{2\sin\theta}\ \theta\cos\theta$

$\frac{dx}{d\theta} = a\theta\cos\theta$

$\frac{dy}{du} = a$ $\quad y = au$

$\frac{dy}{du} = a$

$u = \sin\theta - \theta\cos\theta$

$u = \cos\theta - \cos\theta - \theta\sin\theta$

$u = \theta\sin\theta$

$\frac{dy}{d\theta} = a\theta\sin\theta$

$\frac{dy}{dx} = \frac{\cancel{a\theta}\sin\theta}{\cancel{a\theta}\cos\theta}$

$\frac{dy}{dx} = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$a(\cos\theta + \cancel{\theta} \cos'\theta \sin\theta)$

$$\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta)$$

$$= -a\sin\theta + a\theta\cos\theta + a\sin\theta$$

$$\frac{dx}{d\theta} = a\theta\cos\theta$$

$$y = a(\sin\theta - \theta\cos\theta)$$

$$\frac{dy}{d\theta} = a[\cos\theta - (-\sin\theta)\ \cancel{\theta}\ \cos\theta]$$

$$a[\cos\theta + a\theta\sin\theta - a\cos\theta]$$

$$\frac{dy}{d\theta} = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{1}{\frac{dx}{d\theta}}$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$$

$$\frac{dy}{dx} = \frac{\cancel{a\theta}\sin\theta}{\cancel{a\theta}\cos\theta}$$

$$= \tan\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(\tan\theta)$$

$$\frac{dy}{dx} = \frac{d}{d\theta}(\tan\theta) \cdot \frac{d\theta}{dx}$$

$$\sec^2\theta \cdot \frac{1}{2\theta\cos\theta}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2\theta\cos^3\theta}$$

DIFFERENTIATION APPLICATION OR

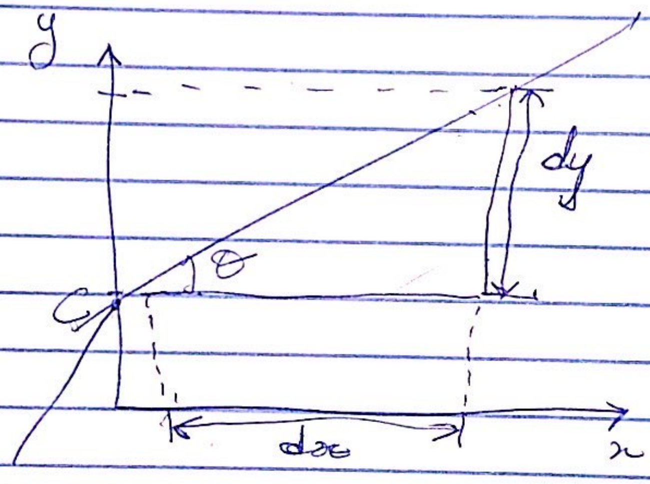
EQUATION OF A STRAIGHT LINE

The basic ~~eq~~ equation of a straight line is given by $y = mx + c$

C = intercept on the y axis

m = Gradient = $\frac{\Delta y}{\Delta x}$

Graphically,



intercept

NOTE If x and y Scaling are identical then Change in y

$y = mx + c$

For Point (a) $x = 3, y = 2$

$2 = 3m + c \longrightarrow (1)$

For Point 2 $x = -2, y$

$1 = -2m + c$

Find the

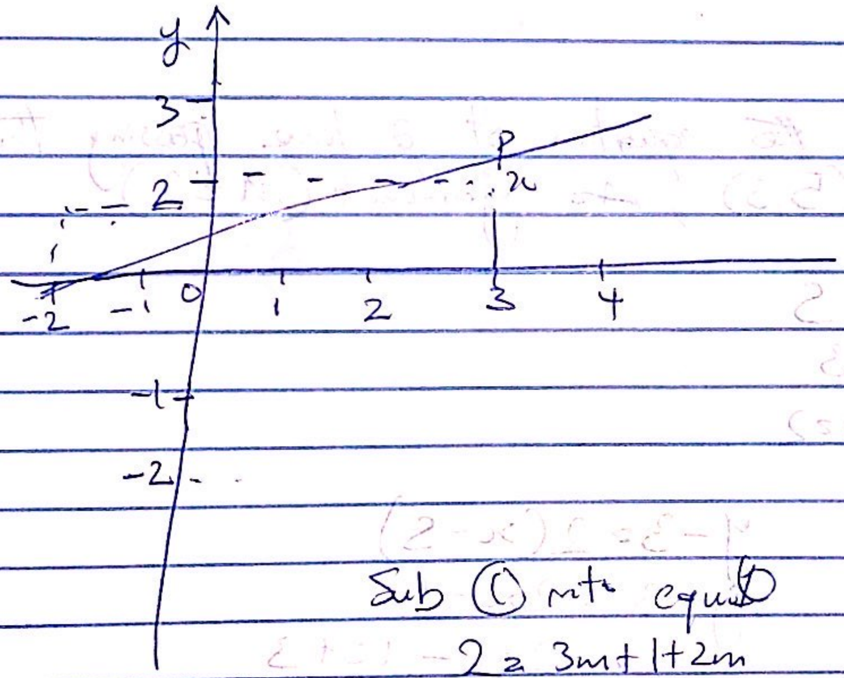
Case I

(8) P, $x = 3, y = 2$

& Q = $x = -2, y = 1$

Ex-1

P(3, 2) and Q(-2, 1)



Sub (1) into equ (2)

$2 = 3m + 1 + 2m$

$2 = 5m + 1$

$5m = 2 - 1$

$m = \frac{1}{5}$

$C = 1 + 2(\frac{1}{5}) = 1 + \frac{2}{5}$

$C = \frac{5+2}{5} = \frac{7}{5}$

$y = mx + c$

$5y = x + 7$

Case two

Sometimes we are giving the gradient m of a Straight line passing through a giving Point (x_1, y_1) and we are required to find its equation

In Such case this is used

$$y - y_1 = m(x - x_1)$$

Example 2

Find the equation of a line passing through the point $(5,3)$, ~~the~~ Gradient $(m = 2)$

$x_1 = 5$

$y_1 = 3$

$m = 2$

$$y - 3 = 2(x - 5)$$

$$y - 3 = 2x - 10$$

$$y = 2x - 10 + 3$$

$$y = 2x - 7$$

$m = 2$

$c = -7$

Case 3

If m and m_1 represent the gradient of two lines perpendicular to each other then, the product $mm_1 = -1$ Therefore $m_1 = \frac{-1}{m}$

$y = mx + c$ — First equation

m — gradient of the first

$y = m_1 x + c_1$ — 2nd equation

m_1 — gradient of the 2nd equ

$mm_1 = -1$ They are perpendicular ($\perp$)

Example 3

Which of this pairs of Straight lines are perpendicular to each other

a $y = 3x - 5$ and $3y = x + 2$

$y = 3x - 5$ $\quad y = \frac{x}{3} + \frac{2}{3}$

$m = 3$

$m_1 = \frac{1}{3}$ $\quad mm_1 = -1$

$\cancel{3} \times \frac{1}{\cancel{3}} = 1$

$mm_1 \neq -1$ They not Perpendicular

b $y - 3x - 2 = 0$ and $3y + x + 9 = 0$

$y = 3x + 2$ $\quad 3y = -x - 9$

$y = \frac{-x}{3} - 3$

$$mm_1 = -1$$
$$3 \times -\frac{1}{3} = -1$$

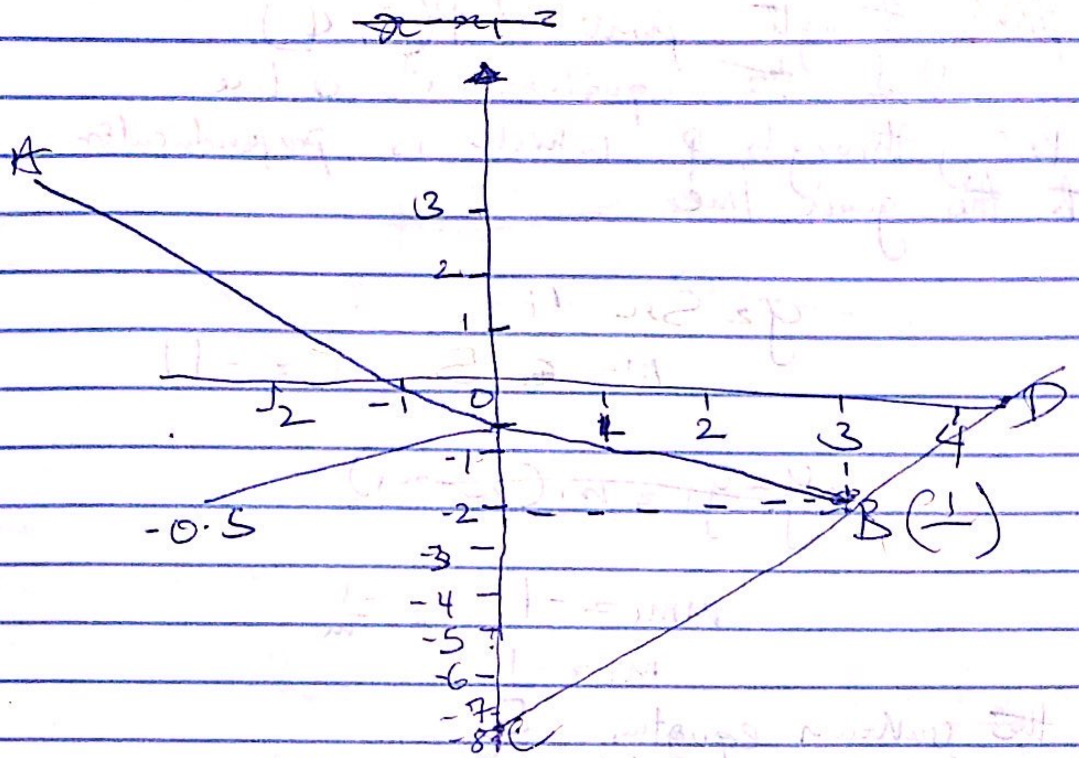
$\therefore$ They are perpendicular

6 ii) $y = 3x + 2$
$m = 3$

$3y = -x - 9$
$y = -\frac{1}{3}x \times -3$

A straight line AB passes through point $P(3, -2)$ with gradient $m = -\frac{1}{2}$ find its equation and also the equation of the line CD

$$y - y_1 = m(x - x_1)$$
$$\cancel{y+2 =}\ y - (-2) = -\frac{1}{2}(x - 3)$$
$$y + 2 = \frac{-x + 3}{2}$$
$$2y + 4 = -x + 3$$
$$2y = -x + 3 - 4$$
$$2y = -x - 1$$
$$y = -\frac{x}{2} - \frac{1}{2}$$
$$m = -\frac{1}{2} \quad c = -\frac{1}{2}$$
$$mm_1 = -1$$
$$m_1 = \frac{-1}{m} \qquad m_1 = -1 \times \frac{-2}{1}$$
$$m_1 = 2$$



$Mm_1 = -1$

$m =$ Gradient of line AB

$m_1 =$ Gradient of line CD

$-\frac{1}{2} \cdot m_1 = -1$

$-m_1 = -2$

$m_1 = 2$

line CD passes through $P(\overset{x_1}{3}, \overset{y_1}{-2})$

$y - y_1 = m_1(x - x_1)$

$y + 2 = 2(x - 3)$

$y + 2 = 2x - 6$

$y = 2x - 8$

$m = 2, \quad C = -8$

$m =$

$mm_1 = -1$

$m_1 = \frac{-1}{m}$

$m_1 = \frac{-1}{-\frac{1}{2}}$

$m_1 = -1 \times \frac{-2}{1}$

$m_1 = +2$

If a straight line equation $y = 5x - 11$ passes through point $P(3, 4)$ find the equation of a line passing through P which is perpendicular to the given line

$$y = 5x - 11$$

$$m = 5 \quad C = -11$$

$$\cancel{y - y_1 = m(x - x_1)}$$

$$mm_1 = -1 \quad = \frac{-1}{m}$$

$$m_1 = \frac{-1}{5}$$

For the unknown equation passing through point, $P(3, 4)$ $m_1 = \frac{-1}{5}$

$$y - 4 = \frac{-1}{5}(x - 3)$$

$$5y - 20 = -x + 3$$

$$5y = -x + 3 + 20$$

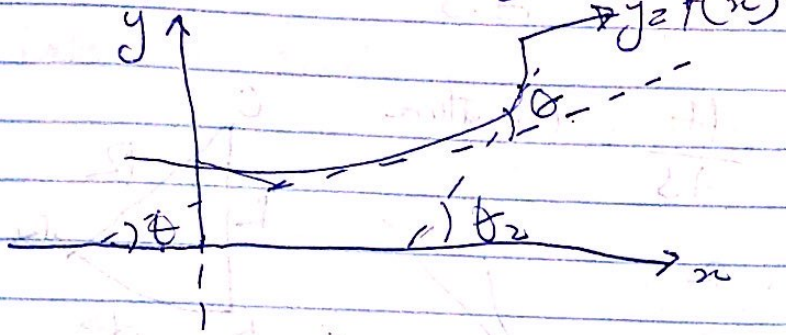
$$5y = -x + 23$$

$$y = \frac{-x}{5} + \frac{23}{5}$$

$$m_1 = -\frac{1}{5} \quad C_1 = \frac{23}{5}$$

Curvature

Curvature is concerned with how quickly the curve is changing direction in the neighbourhood of the point

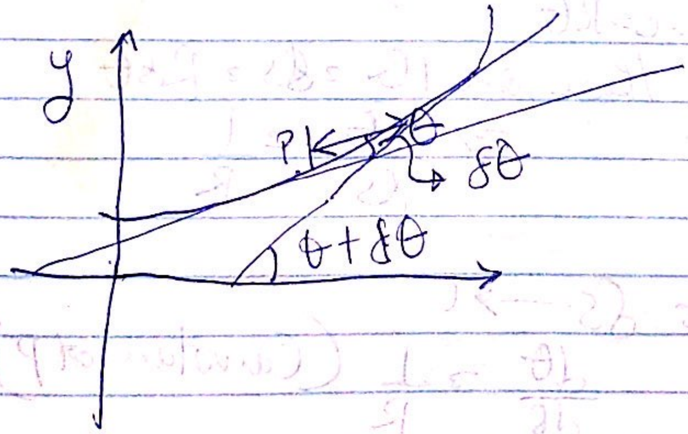


Consider the change in direction of a curve, $y = f(x)$ between points P and Q. The direction of a curve is measured by the gradient of the tangent

Hence Gradient at $P = \tan\theta_1 = \left\{\frac{dy}{dx}\right\}$

Gradient at $Q = \tan\theta = \left\{\frac{dy}{dx}\right\}$

$\therefore \theta = \theta_2 - \theta_1$



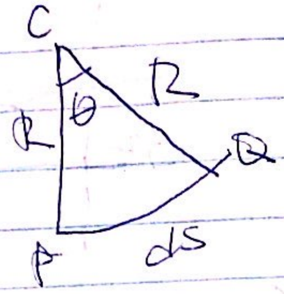
The Average rate of direction change from P to Q $=$ $\dfrac{\text{Change in direction from P to Q}}{\text{length of arc PQ}}$

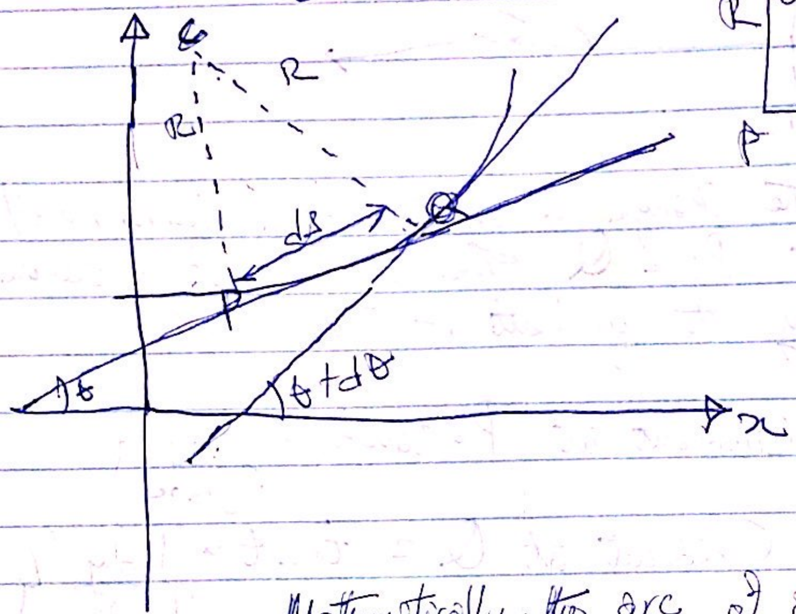
Average Curvature $= \dfrac{\delta\theta}{\delta s}$

If Point Q moves down towards P. i.e $\delta s \to 0$

$$\Rightarrow \frac{\delta\theta}{\delta S} = \frac{d\theta}{dS}$$

$$\frac{d\theta}{dS} = \text{Curvature}$$





Mathematically, the arc of a circle of radius, R which substends at angle θ at the Centre is given by, arc = Rθ

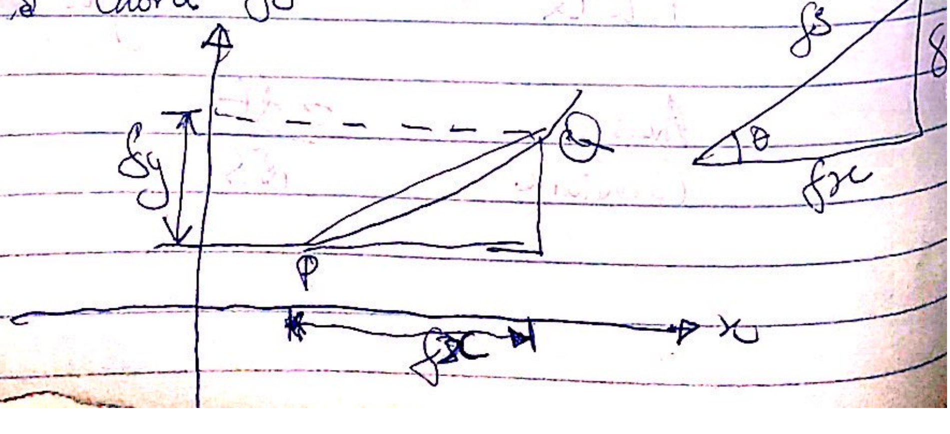
Hence arc PQ = $\delta S = R\delta\theta$

$$\Rightarrow \frac{\delta\theta}{\delta S} = \frac{1}{R}$$

With $\delta S \longrightarrow 0$

$$\frac{d\theta}{dS} = \frac{1}{R} \quad (\text{Curvature at } P)$$

Consider the arc PQ, with points PQ conn... by a Chord δS



with $\delta s \longrightarrow 0$

$\tan\theta = \frac{dy}{dx} \longrightarrow 0, \quad \cos\theta = \frac{dx}{ds}$

Differentiate equation (1) w.r.t 's'

$$\frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d}{ds}(\tan\theta)$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{ds} = \frac{d(\tan\theta)}{d\theta}\cdot\frac{d\theta}{ds}$$

$$\frac{d^2y}{dx}\cos\theta = \sec^2\theta\frac{d\theta}{ds}$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2\theta}{\cos\theta}\frac{d\theta}{ds}$$

$$\frac{d^2y}{dx^2} = \sec^3\theta\frac{d\theta}{ds}$$

$$\sec^3\theta = (\sec^2\theta)^{3/2}$$

$$\cancel{x^{3} = (x^2)^{\frac{3}{2}}}$$

$$\frac{d^2y}{dx^2} = (\sec^2\theta)^{3/2}\frac{d\theta}{ds}$$

From trigonometric function

$$\sec^2\theta = 1+\tan^2\theta$$

$$\frac{d^2y}{dx^2} = (1+\tan^2\theta)^{3/2}\frac{d\theta}{ds}$$

$$\frac{d^2y}{dx^2} = \left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}\frac{d\theta}{ds}$$

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$$\frac{d\theta}{ds} = \frac{\frac{d^2y}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{1}{R}$$

$$\boxed{R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}}$$

Find the radius of Curvature model

Example 1

Find the radius of Curvature for the hyperb
$xy = 4$ point $(x=2; y=2)$

Solution

$xy = 4$, $y = \frac{4}{x}$

$\cancel{\frac{dy}{dx}} = 4 \quad y = \frac{4}{x} = 4x^{-1}$

$y = 4x^{-1}$

$\frac{dy}{dx} = -4x^{-2}$

$\left\{\frac{dy}{dx}\right\} \; x=2 \quad y=2$

$-4(2)^{-2} = \frac{-4}{2^2} = -1$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(-4x^{-2}\right)$$

$$= 8x^{-3}$$

$$\left\{\frac{d^2y}{dx^2}\right\} = 8(2)^{-3}$$

$$= 1$$

$$R = \frac{\left[1 + (-1)^2\right]^{3/2}}{1} = \frac{(2)^{3/2}}{1} = (\sqrt{2})^3 = 2\sqrt{2}$$

$= 2.828$ Units.

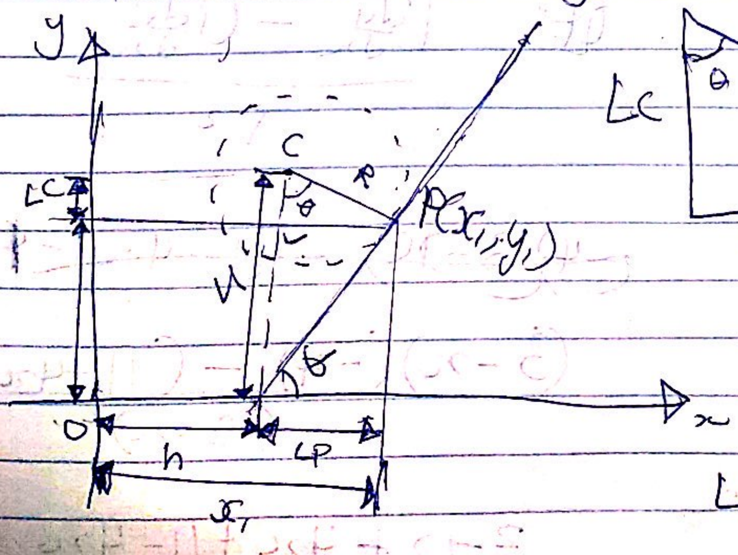
Assignment

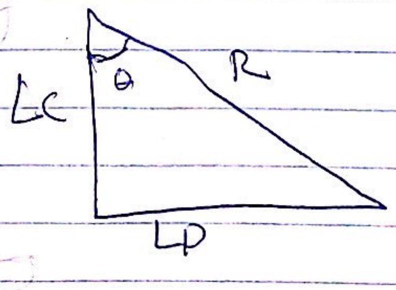
If $x = \theta - \sin\theta$, $y = 1 - \cos\theta$

find R when $\theta = 60° = \frac{\pi}{3}$

Centre of Curvature

If the position of the Centre of the Circle of Curvature for the Point $P(x_1, y_1)$





$$\sin\theta = \frac{LP}{R}$$

$$LP = R\sin\theta$$

$$\cos\theta = \frac{LC}{R}$$

$$LC = R\cos\theta$$

If the Centre C is point (h, k)

$h = x_1 - LP \Rightarrow h = x_1 - R\sin\theta$

$k = y_1 + LC \Rightarrow k = y_1 + R\cos\theta$

~~C(h, k)~~ $C = (h, k)$

$\tan\theta = \frac{dy}{dx}$

$\theta = \tan^{-1}\left(\frac{dy}{dx}\right)$

Example 2

Find the Radius of Curvature and the Co-ordinates of Centre of Curvature of the Curve $y = \frac{11-4x}{3-x}$ at point $(2, 3)$

Solution

$\frac{dy}{dx} = \quad \frac{du}{dx} = -4 \qquad \frac{dv}{dx} = -1$

$\frac{V\frac{du}{dx} - U\frac{dv}{dx}}{V^2}$

$\cancel{(-4)(3-x) - (11-4x)-1}$

$\frac{(3-x)(-4) - (11-4x)(-1)}{(3-x)^2}$

$= \frac{-12 + 4x + 11 - 4x}{(3-x)^2}$

$\frac{dy}{dx} = \frac{-1}{(3-x)^2}$

$$\left\{\frac{dy}{dx}\right\} = \frac{-1}{(3-2)^2} = \frac{-1}{1} = -1$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{-1}{(3-x)^2}\right] = \frac{d}{dx}\left[-(3-x)^{-2}\right]\frac{dy}{dx}$$

$$= 2(3-x)^{-3}\cdot(-1)$$
$$= -2(3-x)^{-3}$$

Solution

$$\left\{\frac{d^2y}{dx^2}\right\} = -2(3-2)^{-3}$$

$$= -2(1)^{-3}$$

$$= -2$$

$$R = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+(-1)^2\right]^{\frac{3}{2}}}{-2}$$

$$= \frac{(2)^{3/2}}{-2} = \frac{(\sqrt{2})^3}{-2} = \frac{\cancel{2}\sqrt{2}}{-\cancel{2}} = -\sqrt{2}$$

$$R = -\sqrt{2} \text{ units}$$

Solution

$$\tan\theta = \frac{dy}{dx} = -1$$

$$\theta = \tan^{-1}(-1) = -45^\circ$$

$$\theta = -45^\circ$$

$h = 2 - [-\sqrt{2}\sin(-45^\circ)]$

$2 - \left(-\sqrt{2} \times -\frac{1}{\sqrt{2}}\right)$

$2 - (1) = 2 - 1$

$h = 1$

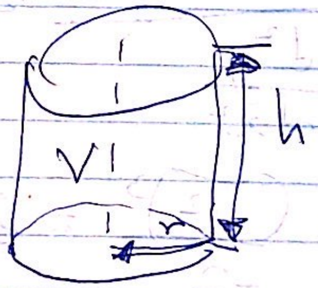
$k = 3 + [-\sqrt{2}\cos(-45^\circ)]$

$= 3 + \left[-\sqrt{2} \times \frac{1}{\sqrt{2}}\right] = 3$

$k = 3 - 1$, $k = 2 \Rightarrow$ Centre of Curve

PARTIAL DIFFERENTIATION

The Volume, V of a Cylinder of radius, r ~~and~~ height, h is given by $V = \pi r^2 h$



$$\left[\frac{dV}{dh}\right]_{r = \text{constant}} = \frac{\partial V}{\partial h} = \pi r^2$$

$$\left[\frac{dV}{dr}\right]_{h = \text{constant}} = \frac{\partial V}{\partial r} = 2\pi r h$$

Consider a function

$$Z = x^2y^3$$

$$\frac{\partial Z}{\partial x} = 2xy^3$$

$$\frac{\partial Z}{\partial y} = 3x^2y^2$$

Example I

$$U = x^3 + y^2 - 2x^2y$$

find $\frac{du}{dx}, \frac{du}{dy}$

$$\frac{du}{dx} = 3x^2 + 0 - 4xy$$

$$3x^2 - 4xy$$

$$\frac{du}{dy} = \cancel{3x^2 + 5}\ 0 + 3y^2 - 2x^2$$

$$= 3y^2 - 2x^2$$

Ex 2

If $Z = \sin(3x+2y)$

find $\frac{dZ}{dx}$ and $\frac{dZ}{dy}$

$$\cancel{\frac{\partial Z}{\partial x} = \cos(3x+2y)\frac{\partial}{\partial x}}\ \frac{\partial}{\partial}$$

$$\frac{dz}{dx} = \cos(3x+2y)\cdot\frac{\partial}{\partial x}(3x+2y)$$

$$\cos(3x+2y)\cdot 3$$

$$= 3\cos(3x+2y)$$

$$\frac{\partial Z}{\partial y} = \cos(3x+2y)\frac{d}{dy}(3x+2y)$$

$$= 2\cos(3x+2y)$$

$Z = 2^{-4x} - \cos 2y$

Assignment

$Z = (2e^{-4x} - \cos 2y)(x^3 + 4y)\ \frac{dz}{dx}, \frac{dy}{dx}$

SECOND ORDER

$Z = 3x^2 + 4xy - 5y^2$

$\frac{dz}{dx} = 6x + 4y$

$\frac{dz}{dy} = 4x - 10y$

$\frac{d}{dx}\left(\frac{dz}{dx}\right) = \frac{d^2z}{dx^2} = 6$

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = 4$

$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = -10$

$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = 4$

If $V = f(x^2 + y^2)$

Show that $x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x} = 0$

$\frac{\partial V}{\partial x} = f'(x^2 + y^2) \cdot \frac{\partial}{\partial x}(x^2 + y^2)$

$f'(x^2+y^2).2x$

$\frac{dV}{dx} = 2x f'(x^2+y^2)$

$\frac{dv}{dy} = f'(x^2+y^2).\frac{d}{dy}(x^2+y^2)$

$= 2y f'(x^2+y^2)$

$= 2xy + f'(x^2+y^2)$

$y\frac{dV}{dx} = 2xyf'(x^2+y^2)$

$x\frac{dV}{dy} - y\frac{dv}{dx} = 0$

Assignment

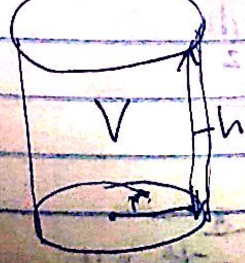
(1) $Z = x+y - y\cos x$

Show that ~~partial~~

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

SMALL INCREMENTS

$V = \pi r^2 h$



$\frac{dv}{dr} = 2\pi rh$

$\frac{dv}{dh} = \pi r^2$

If both r and h changes simultaneously

$\Rightarrow r = r + \delta r,\ h = h + \delta h$

$V = V + \delta V$

The new Volume, V

$V + \delta V = \pi(r+\delta r)^2(h+\delta h)$

$= \pi(r^2 + 2r\delta r + (\delta r)^2)(h+\delta h)$

$= \pi(r^2h + 2rh\delta r + h(\delta r)^2 + r^2\delta h + 2r\delta r\delta h + (\delta r)^2\delta h)$

Subtract $V = \pi r^2 h$ from both Sides

$\delta V = \pi(2rh\delta r + h(\delta r)^2 + r^2\delta h + 2r\delta r\delta h + (\delta r)^2\delta h)$

$\delta r, \delta h$ are very small ~~entiti~~ entities

$V \approx 2\pi rh\delta r + \pi r^2\delta h$

$$\boxed{\delta V = \frac{\delta V}{\delta r}\delta r + \frac{\delta V}{\delta h}\delta h}$$

$h = x - R\sin\theta$

$k = y + R\cos\theta$

A Cylinder of $r = 5cm$ $h = 10cm$. Find the approximate increase in volume when r increases by 0.2cm and h increases by (-0.1)cm

Solution

$V = \pi r^2 h$

$\delta h = \frac{\delta V}{\delta r}$ or $\frac{\delta V}{\delta h} \delta h$

$\delta r = 0.2$, $\delta h = -0.1$

$\frac{\delta V}{\delta r} = 2\pi r = 2\pi(5)(10)$

$= 100\pi$

$\frac{\delta V}{\delta h} = \pi r^2 = \pi 5^2 = 25\pi$

$\delta V = 100\pi \cdot 0.2 + 25\pi \cdot (-0.1)$

$= 2.5\pi = 17.5\pi$

$\delta V = 54.98 cm^3$

17 $y = \frac{w s^3}{d^4}$ $\frac{ws^3}{d^4}$

Ans $\boxed{-11y}$

Find the percentage increase in y when w increases by 2% and S decreases by 3% d increases by 1%

$y = \frac{ws^3}{d^4}$ $\left[dy = \frac{dy}{dw} \cdot dw + \frac{dy}{ds} \cdot ds + \frac{dy}{dd} \cdot dd\right]$

$\ln y = \ln w + \ln s^3 - \ln d^4$

$\ln w = \frac{1}{w} \times 1$

$\ln y = \ln w$

$\frac{1}{y} \frac{dy}{dw} = \frac{1}{w}$

$\frac{dy}{dw} = \frac{1}{w} \times y$

$= \frac{1}{w} \times \frac{w s^3}{d^4}$

$= \frac{s^3}{d^4}$

$\ln y = \ln s^3$

$\frac{1}{y} \frac{dy}{ds} = \frac{1}{s^3} 3s^2$

$\frac{dy}{ds} = \frac{3}{s} \times y$

$= \frac{3}{s} \times \frac{ws^3}{d^4}$

$\frac{dy}{ds} = \frac{3ws^2}{d^4}$

$\ln y = \ln d^4$

$\frac{1}{y} \frac{dy}{dd} = \frac{1}{d^4} 4d^3 = \frac{4}{d} \times y$

$\frac{dy}{dd} = -\frac{4}{d} \times \frac{ws^3}{d^4} = \frac{-4ws^3}{d^5}$